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FIRST BOREL CLASS SETS IN BANACH SPACES AND THE ASYMPTOTIC-NORMING PROPERTY

BY

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ABSTRACT

The Radon–Nikodým property in a separable Banach space X is related to the representation of X as a weak* first Borel class subset of some dual Banach space (its bidual X^{**}, for instance) by well known results due to Edgar and Wheeler [8], and Ghoussoub and Maurey [9, 10, 11]. The generalizations of those results depend on a new notion of Borel set of the first class "generated by convex sets" which is more suitable to deal with non-separable Banach spaces. The asymptotic-norming property, introduced by James and Ho [13], and the approximation by differences of convex continuous functions are also studied in this context.

1. Introduction

This paper is devoted to showing several connections between geometrical properties of a Banach space and certain kinds of descriptive sets. Dealing with non-metrizable topologies, as the weak topology of a Banach space, new types of Borel subsets appear, even of the first class, that is, obtained by just one countable operation. We say that a subset A of a topological space V is $(\mathcal{F}\wedge\mathcal{G})_\sigma$ (resp. $(\mathcal{F}\vee\mathcal{G})_\delta$) if there are closed subsets F_n and open subsets G_n of V such that $A=\bigcup_{n=1}^{\infty}(F_n\cap G_n)$ (resp. $A=\bigcap_{n=1}^{\infty}(F_n\cup G_n)$). Clearly, a subset is $(\mathcal{F}\wedge\mathcal{G})_\sigma$ if, and only if, its complement is $(\mathcal{F}\vee\mathcal{G})_\delta$.

We shall consider a Banach space X as a topological subspace of its bidual X^{**} endowed with the weak* topology. Jayne, Namioka and Rogers [14] proved

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that if a Banach space X is a Borel subset of X^{**}, then X has a certain topological property called σ-fragmentability. It is natural to expect that X will have stronger properties if it is of lower Borel class in X^{**}. Indeed, Edgar and Wheeler showed [8] that X has the point of continuity property (PCP) if X is a $(\mathcal{F} \vee \mathcal{G})_\delta$ subset of X^{**} and the converse is true if X is separable. Recall that a map has the **point of continuity property** if its restriction to any nonempty closed subset has a point of continuity and a Banach space X has the point of continuity property (PCP) if the identity map on B_X from the weak to the norm topology has it. If $X^{**} \setminus X = \bigcup_{n=1}^{\infty} K_n$ where every K_n is convex and weak* compact, then X has the Radon–Nikodým property (RNP) [8]. This is far from being a characterization, because if a separable Banach space X is a $\mathcal{G}_\delta$ subset of X^{**}, then X^* is also separable. Ghoussoub and Maurey [9] proved the following: *A separable space X has the RNP if, and only if, there is a separable Banach space Y such that $X \subset Y^*$ (isomorphically) and $Y^* \setminus X = \bigcup_{n=1}^{\infty} K_n$ where every K_n is convex and weak* compact.*

We shall consider a particular subclass of the $(\mathcal{F} \wedge \mathcal{G})_\sigma$ sets in a topological linear space (V, τ). A subset $D \subset V$ is said to be a **$(\mathcal{C} \setminus \mathcal{C})_\sigma$ set with respect to τ** if $D = \bigcup_{n=1}^{\infty} (A_n \setminus B_n)$, where A_n and B_n are convex τ-closed subsets of V. In case τ is the norm topology, we just say $(\mathcal{C} \setminus \mathcal{C})_\sigma$ set. Using the notion of $(\mathcal{C} \setminus \mathcal{C})_\sigma$ set we are able to give the following characterization of the RNP.

THEOREM 1.2: *A separable Banach space X has the RNP if, and only if, $X^{**} \setminus X$ is a $(\mathcal{C} \setminus \mathcal{C})_\sigma$ set with respect to the weak* topology.*

A well known result of Edgar [7] connecting renorming theory and Borel structure establishes that, in a Banach space with a Kadec norm, every norm open set is an $(\mathcal{F} \wedge \mathcal{G})_\sigma$ set with respect to the weak topology. Recall that the norm $\|.\|$ of X is said to be **locally uniformly rotund** (LUR) if for every $x, x_k \in X$, such that $\lim_k \|x_k\| = \|x\|$ and $\lim_k \|x + x_k\| = 2\|x\|$, then $\lim_k \|x - x_k\| = 0$. A LUR norm is Kadec; see [6] for this and further information.

THEOREM 1.2: *Every norm open subset of a Banach space X is a $(\mathcal{C} \setminus \mathcal{C})_\sigma$ set if, and only if, X has an equivalent locally uniformly rotund norm.*

Let V be a topological space and d a metric on V not necessarily related to the topology of V. We say that V is **fragmented** by d if every nonempty subset of V has a nonempty relatively open subset of arbitrarily small diameter. Weakly compact subsets of a Banach space are norm fragmented [21]. If X is a Banach space and $Y \subset X^*$ is a norming subspace, then there is a canonical embedding of X into Y^*. In this case, the topology $\sigma(X, Y)$ is the restriction of the weak* topology of Y^*. We shall always regard X as a subset of Y^*. Further, after a suitable renorming, we may assume that $B_{Y^*} = \overline{B_X}^{w^*}$. The following results generalize [8, Theorem 4.13] and part of [9, Theorem III.1].

THEOREM 1.3: *Let X be a Banach space and $Y \subset X^*$ a norming subspace. Assume that the $\sigma(X,Y)$-compact subsets of X are fragmented by the norm and X is a $(\mathcal{F} \vee \mathcal{G})_\delta$ subset of Y^* in the weak* topology. Then the identity map on B_X from the $\sigma(X,Y)$-topology to the norm has the point of continuity property.*

THEOREM 1.4: *Let X be a Banach space and $Y \subset X^*$ a norming subspace. Assume that the $\sigma(X,Y)$-compact subsets of X are fragmented by the norm and $Y^* \setminus X$ is $(\mathcal{C} \setminus \mathcal{C})_\sigma$ subset of Y^* in the weak* topology. Then X has the RNP.*

For a separable Banach space Y the following three conditions are equivalent: (1) Y^* *has the RNP*; (2) Y^* *is separable*; (3) Y^* *has an equivalent dual norm* $\|.\|$ *such that* $w^*\text{-}\lim_n y_n^* = y^*$ *and* $\lim_n \|y_n^*\| = \|y^*\|$ *implies that* $\lim_n \|y_n^* - y^*\| = 0$. Bourgain and Delbaen [2], and McCartney and O'Brien [18], built separable Banach spaces having the RNP and isomorphic to no subspace of a separable dual Banach space. However, separable Banach spaces with RNP could be characterized by a renorming property in the spirit of condition (3) above. In this sense, James and Ho [13] introduced the asymptotic-norming property (ANP) and showed that it implies the RNP. The equivalence between the RNP and the ANP for separable spaces was established by Ghoussoub and Maurey [11]. Although we leave the technical definition of the ANP for the next paragraph, what they actually proved reads as follows: *A separable Banach space X has the RNP if, and only if, there is a separable Banach space Y such that X is isomorphic to a subspace of Y^* and the following property is verified: for any sequence $(x_n) \subset X$, such that $w^*\text{-}\lim_n x_n = y^* \in Y^*$ and $\lim_n \|x_n\| = \|y^*\|$, then* $\lim_n \|x_n - y^*\| = 0$. A main argument used in [11] to prove the previous equivalence motivated the authors to introduce the concept of strong $\mathcal{H}_\delta$ subset. We shall take advantage of their idea to define a similar notion. Let $(V, \|.\|)$ be a normed space and let τ be a vector topology on V and denote by d the induced distance between sets. A subset $D \subset V$ is said to be a **strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$ set** (with respect to τ) if $D = \bigcup_{n=1}^{\infty} (A_n \setminus B_n)$ where A_n and B_n are convex τ-closed subsets of V such that $d(X \setminus D, A_n \setminus B_n) > 0$ for every $n \in \mathbb{N}$.

We do not need to deal with all the different ANP's introduced in [13], so we shall just use the weakest one along the paper, named ANP-III there. Given a norming subset $\Phi \subset B_{X^*}$, we say that a sequence $(x_n) \subset S_X$ is asymptotically normed by Φ, if for every $\varepsilon > 0$, there is $y \in \Phi$ and $N \in \mathbb{N}$, such that

$y(x_n) > 1 - \varepsilon$ for all $n \geq N$. We say that a Banach space X has the Φ-ANP if $\bigcap_{n=1}^{\infty} \overline{conv}^{\|\cdot\|}(\{x_m : m \geq n\}) \neq \emptyset$ for every sequence (x_n) asymptotically normed by Φ. We say that X has the ANP if it has the Φ-ANP for some norming subset $\Phi \subset B_{X^*}$. The characterization of the ANP obtained by Hu and Lin [12], which avoids asymptotically normed sequences, is a main tool to prove the following:

THEOREM 1.5: *A Banach space X has the ANP with some equivalent norm if, and only if, $X^{**} \setminus X$ is a strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$ subset of X^{**} with respect to the weak* topology.*

In relation with this result, we shall discuss conditions on a Banach space X implying that $X^{**} \setminus X$ is $(\mathcal{C} \setminus \mathcal{C})_\sigma$ or strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$ into (X^{**}, w^*). The separable case is generalized in a quite natural way in terms of Szlenk type indices; see Theorem 3.5. A different kind of example is provided by Banach spaces isomorphic to dual Banach spaces having a LUR norm, Theorem 4.4.

Measurability with respect to $(\mathcal{C} \setminus \mathcal{C})_\sigma$ sets is related to approximation by differences of convex functions; see Proposition 2.1. We give the following extension of a recent result of Cepedello-Boiso [5]:

THEOREM 1.6: *A function $h: X \to \mathbb{R}$ defined on a Banach space is a pointwise limit of a sequence of differences of convex continuous functions if the sets $h^{-1}(r, +\infty)$ and $h^{-1}(-\infty, r)$ are strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$ for all $r \in \mathbb{R}$.*

The rest of the paper is devoted to proving the results listed in this introduction as a part of a common frame. In section 2 we study $(\mathcal{C} \setminus \mathcal{C})_\sigma$ and strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$ subsets of a Banach space. In section 3 we deal with Banach spaces representable as first Borel class subsets of a dual space. The Banach spaces which embed as strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$ subsets of a dual space are studied in section 4 in relation with the ANP. Although the paper deals mainly with non-separable Banach spaces, we show in Proposition 4.7 how our techniques apply to provide a new proof of the equivalence between RNP and ANP for a separable Banach space.

2. Differences of convex sets

The first result of the section shows that $(\mathcal{C} \setminus \mathcal{C})_\sigma$ sets appear naturally when one deals with differences of convex functions.

PROPOSITION 2.1: *Let X be a Banach space and $h: X \to \mathbb{R}$ a function which is uniform limit of differences of convex functions. Then both $h^{-1}(-\infty, r)$ and $h^{-1}(r, +\infty)$ are $(\mathcal{C} \setminus \mathcal{C})_\sigma$ subsets of X for every $r \in \mathbb{R}$.*

Proof: First assume that $h = f_1 - f_2$ where f_1 and f_2 are convex. Clearly, we can restrict ourselves to examine the set $h^{-1}(0, +\infty)$. We have

$$h^{-1}(0,+\infty) = \bigcup_{r<s}\{x \in X: f_1(x) > s, f_2(x) \le r\}$$
$$= \bigcup_{r<s}(\{x \in X: f_2(x) \le r\} \setminus \{x \in X: f_1(x) \le s\})$$

where the indices r and s are rational. Thus $h^{-1}(0, +\infty)$ is expressed as a countable union of differences of convex closed sets.

Let h be uniform limit of a sequence (h_n) of differences of convex functions. Changing $h_n(x)$ by $h_n(x) - \|h - h_n\|_\infty$ we may assume that $h_n \le h$. In that case, $h^{-1}(r, +\infty) = \bigcup_{n=1}^\infty h_n^{-1}(r, +\infty)$ for any $r \in \mathbb{R}$. ∎

Remark 2.2: A similar result does not hold for pointwise limits even in case of $X = \mathbb{R}$. The Cantor set C is not $(\mathcal{C} \setminus \mathcal{C})_\sigma$ and it is quite easy to put the characteristic function of C as a pointwise limit of differences of convex functions.

For a subset $A \subset X$, we shall denote A^o the norm interior of A. Given $Z \subset X^*$ a norming subspace and $A \subset X$ convex and $\sigma(X, Z)$-closed, we shall consider the set

$$\mathcal{B}[A, r] = \overline{A + rB_X}^{\sigma(X,Z)}.$$

It is easy to see that $\bigcap_{r>0} \mathcal{B}[A, r] = A$. In the following, we shall assume that the norming subspace $Z \subset X^*$ is 1-norming. To do that, just take the norm on X which has as unit ball $\overline{B_X}^{\sigma(X,Z)}$. The set of all $\sigma(X, Z)$-open halfspaces of X will be denoted by $\mathbb{H}(Z)$.

LEMMA 2.3: *Let X be a Banach space and $Z \subset X^*$ a norming subspace. For a given subset E of X the following conditions are equivalent:*

(i) *$X \setminus E$ is a strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$ subset of X with respect to $\sigma(X, Z)$.*
(ii) *There are sequences of convex $\sigma(X, Z)$-closed sets (A_n) and (B_n) with nonempty norm interior such that $X \setminus E = \bigcup_{n=1}^\infty (A_n^o \setminus B_n)$.*
(iii) *There is a sequence of convex $\sigma(X, Z)$-closed subsets (A_n) such that for every $x \in X \setminus E$, there is $H \in \mathbb{H}(Z)$ verifying that $x \in A_n \cap H$ and $d(A_n \cap H, E) > 0$.*
(iv) *There is a sequence of convex $\sigma(X, Z)$-closed subsets (A_n) with nonempty norm interior such that for every $x \in X \setminus E$, there is $H \in \mathbb{H}(Z)$ such that $x \in A_n^o \cap H$ and $A_n \cap H \subset X \setminus E$.*

Proof: (i) ⇒ (iii) Follows easily by the Hahn–Banach theorem.

(iii) $\Rightarrow$ (iv) Let A be a $\sigma(X,Z)$-closed convex set, suppose that $x_0 \in A$ and there is $H \in \mathbb{H}(Z)$ such that $d(A \cap H, E) > 0$. If $H = \{x \in X\colon z(x) > a\}$, then take $b > a$ such that the halfspace $H' = \{x \in X\colon z(x) > b\}$ still contains x_0. Take $0 < r < \inf(d(A \cap H, E), d(H', A \setminus H))$. It is easy to see that

$$\mathcal{B}[A,r] \cap H' \subset \mathcal{B}[A \cap H, r] \subset X \setminus E.$$

Suppose now that (A_n) is a sequence as in statement (iii). The argument above shows that the double sequence $\mathcal{B}[A_n, m^{-1}]$ satisfies statement (iv).

(iv) $\Rightarrow$ (ii) Let B_n be the $\sigma(X,Z)$-closed convex set obtained from A_n removing the $\sigma(X,Z)$-open slices disjoint from E, namely

$$B_n = \{x \in A_n\colon \text{such that } A_n \cap H \cap E \neq \emptyset \text{ whenever } x \in H \in \mathbb{H}(Z)\}.$$

Then we have $X \setminus E = \bigcup_{n=1}^{\infty}(A_n^o \setminus B_n)$. To ensure nonempty norm interiors replace the sequence (B_n) by the double sequence $\mathcal{B}[B_n, m^{-1}]$.

(ii) $\Rightarrow$ (i) Fix $a_n \in A_n^o$ and take

$$A_{n,m} = a_n + (1 - m^{-1})(A_n - a_n).$$

Then we have $d(A_{n,m}, X \setminus A_n^o) > 0$. Define the sets $B_{n,m} = \mathcal{B}[B_n, m^{-1}]$. Since $E \subset (X \setminus A_n^o) \cup B_n$, we deduce that $d(A_{n,m} \setminus B_{n,m}, X \setminus E) > 0$. On the other hand, $X \setminus E = \bigcup_{n,m=1}^{\infty}(A_{n,m} \setminus B_{n,m})$. ∎

THEOREM 2.4: *Let X be a Banach space, let $Z \subset X^*$ be a norming subspace and let $E \subset X$ be a subset. The following properties are equivalent:*

(i) *$X \setminus E$ is a strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$ subset of X with respect to $\sigma(X,Z)$.*

(ii) *There is an equivalent $\sigma(X,Z)$-lower semicontinuous norm $|||.|||$ such that for any $(x_n) \subset E$ and $x \in X$ with $\lim_n |||x_n||| = |||x|||$ and $\lim_n |||x_n + x||| = 2|||x|||$, then $x \in E$.*

Proof: (i) $\Rightarrow$ (ii) By Lemma 2.3, we may assume that $X \setminus E = \bigcup_{n=1}^{\infty}(A_n^o \setminus B_n)$ where the sets A_n and B_n are $\sigma(X,Z)$-closed convex and have nonempty norm interior. Fix interior points $a_n \in A_n$ and $b_n \in B_n$ for every $n \in \mathbb{N}$ and let f_n and g_n be the Minkowski functionals with respect to the points a_n and b_n of the set A_n and B_n, respectively. We define a symmetric convex function F on X by the formula

$$F(x)^2 = \|x\|^2 + \sum_n \alpha_n f_n(x)^2 + \sum_n \beta_n g_n(x)^2 + \sum_n \alpha_n f_n(-x)^2 + \sum_n \beta_n g_n(-x)^2$$

where (α_n) and (β_n) are positive constants taken in such a way to guarantee the uniform convergence on bounded subsets of X of the series, so F is uniformly continuous on bounded sets, and that the absolutely convex set $B = \{x \in X: F(y^*) \leq 1\}$ contains 0 as an interior point. Let $|||.|||$ be the functional of Minkowski of B. Then $|||.|||$ is an equivalent $\sigma(X,Z)$-lower semicontinuous norm on X. It is standard to check that for every sequence $(x_k) \subset B$ with $\lim_k |||x_k||| = 1$, then $\lim_k F(x_k) = 1$.

Suppose that we are given a sequence $(x_n) \subset E$ such that $\lim_n |||x_n||| = |||x|||$ and $\lim_n |||x_n + x||| = 2|||x|||$. We want to show that $x \in E$. Suppose that $x \in X \setminus E$ in order to get a contradiction. Then $x \in A_n^o \setminus B_n$ for some $n \in \mathbb{N}$, and thus $f_n(x) < 1$ and $g_n(x) > 1$. A standard convexity argument [6, Fact 2.3] shows easily that $\lim_k f_n(x_k) = f_n(x)$ and $\lim_k g_n(x_k) = g_n(x)$. For k large enough we should have $f_n(x_k) < 1$ and $g_n(x_k) > 1$, which implies that $x_k \in A_n^o \setminus B_n \subset X \setminus E$, a contradiction.

(ii) $\Rightarrow$ (i) Assume X is endowed with $|||.|||$. First we shall show that for every $x \in X \setminus E$ there is a rational $r > |||x|||$ and $H \in \mathbb{H}(Z)$ such that $x \in \mathcal{B}[0,r] \cap H$ and $\mathcal{B}[0,r] \cap H \subset X \setminus E$. Suppose not and take rational numbers $s_n < |||x||| < r_n$ such that $\lim_n (r_n - s_n) = 0$ and take $H_n \in \mathbb{H}(Z)$ containing x and disjoint from $\mathcal{B}[0,s_n]$. To get a contradiction, take $x_n \in \mathcal{B}[0,r_n] \cap H \cap E$. By the construction, $\lim_n |||x_n||| = |||x|||$ and $\lim_n |||x_n + x||| = 2|||x|||$, thus $x \in E$ which is impossible. If (A_n) is an enumeration of the balls $\mathcal{B}[0,r]$ with $r > 0$ rational, the argument above shows that for every $x \in X \setminus E$, there is $n \in \mathbb{N}$ and $H \in \mathbb{H}(Z)$ such that $x \in A_n^o \cap H$ and $A_n \cap H \subset X \setminus E$. Finally apply Lemma 2.3. ∎

The following result includes Theorem 1.2.

THEOREM 2.5: *For a Banach space X the following conditions are equivalent:*

(i) *X has an equivalent LUR norm.*

(ii) *Every norm open subset of X is a $(\mathcal{C} \setminus \mathcal{C})_\sigma$ set.*

(iii) *Every norm open subset of X is a strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$ set.*

Proof: (iii) $\Rightarrow$ (ii) It is trivial.

(ii) $\Rightarrow$ (i) Let $\mathfrak{B} = \bigcup_{n=1}^\infty \mathfrak{B}_n$ be a basis for the norm topology such that every $\mathfrak{B}_n$ is discrete [15, p. 236]. Take $V_n = \bigcup \mathfrak{B}_n$. By the hypothesis we can find convex closed sets $(A_{m,n})$ and $(B_{m,n})$ such that $V_n = \bigcup_{m=1}^\infty (A_{m,n} \setminus B_{m,n})$. We claim that for every $x \in X$ and $\varepsilon > 0$ there is an open halfspace H and $m, n \in \mathbb{N}$ such that $x \in A_{m,n} \cap H$ and $\operatorname{diam}(A_{m,n} \cap H) < \varepsilon$. Indeed, for some $n \in \mathbb{N}$, there is $W \in \mathfrak{B}_n$ such that $x \in W \subset B(x,\varepsilon)$. Now, for some m we have $x \in A_{m,n} \setminus B_{m,n}$. Let H be an open halfspace containing x and disjoint from $B_{m,n}$. We have

$A_{m,n} \cap H$ is convex and it is contained in V_n. By the discreteness of $\mathfrak{B}_n$ we must have $A_{m,n} \cap H \subset W \subset B(x, \varepsilon)$. The existence of an equivalent LUR norm is now a consequence of the theorem of Moltó, Orihuela and Troyanski [19] (see also [23]).

(i) $\Rightarrow$ (iii) An equivalent LUR norm satisfies condition (ii) of Theorem 2.4 for any norm closed subset E, and thus, every norm open subset is strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$. ∎

Proof of Theorem 1.6: Expressing h as the difference of its positive and negative parts, we may assume that $h \geq 0$. Indeed, if $h = h_+ - h_-$ then $h_+^{-1}(r, +\infty) = X$, $h_+^{-1}(-\infty, r) = \emptyset$ for $r < 0$, and $h_+^{-1}(r, +\infty) = h^{-1}(r, +\infty)$, $h_+^{-1}(-\infty, r) = h^{-1}(-\infty, r)$ for $r \geq 0$ (equalities for h_- are similar). Let (E_n) be an enumeration of the sets of the form $h^{-1}[r, +\infty)$ and $h^{-1}(-\infty, r]$ with r rational. Let $\|.\|_n \leq \|.\|$ be the norm given by Theorem 2.4 for the set E_n. Define an equivalent norm

$$\|x\|^2 = \sum_{n=1}^{\infty} 2^{-n} \|x\|_n^2.$$

We claim that the norm $\|.\|$ has the following property: $\lim_k h(x_k) = h(x)$, whenever $\lim_k \|x_k\| = \|x\|$ and $\lim_k \|x_k + x\| = 2\|x\|$. Take $r > h(x)$ a rational. Clearly, it is enough to show that $h(x_k) < r$ for k large. Suppose not; then $h(x_k) \geq r$ for infinitely many k's. Take n such that $E_n = h^{-1}[r, +\infty)$. By [6, Fact 2.3] we have $\lim_k \|x_k\|_n = \|x\|_n$ and $\lim_k \|x_k + x\|_n = 2\|x\|_n$. This implies $x \in E_n$, and thus $h(x) \geq r$, which is a contradiction.

To express h as a pointwise limit of differences of convex continuous functions we shall use an argument from [5]. Consider

$$\begin{aligned} h_n(x) &= \inf_{y \in X} \{h(y) + n(2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2)\} \\ &= 2n\|x\|^2 - \sup_{y \in X} \{n\|x+y\|^2 - 2n\|y\|^2 - h(y)\}. \end{aligned}$$

Notice that $h_n(x) \leq h(x)$ just putting $y = x$. The last equality expresses h_n as a difference of two convex continuous functions. We claim that $\lim_n h_n(x) = h(x)$. Indeed, take $x_n \in X$ such that

$$h(x_n) + n(2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2) < h_n(x) + n^{-1}.$$

Thus we have

$$\begin{aligned} 0 &\leq 2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2 \\ &< n^{-1}(h_n(x) - h(x_n)) + n^{-2} \\ &\leq n^{-1} h(x) + n^{-2}. \end{aligned}$$

As the last term goes to 0 when n grows, we deduce that $\lim_n 2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2 = 0$, and by [6, Fact 2.3] this is equivalent to $\lim_n \|x_n\| = \|x\|$ and $\lim_n \|x_n + x\| = 2\|x\|$. Using the first part of the proof, we have $\lim_n h(x_n) = h(x)$. On the other hand,

$$h(x_n) - n^{-1} < h_n(x) \leq h(x)$$

and thus $\lim_n h_n(x) = h(x)$. This ends the proof. ∎

Remark 2.6: Theorem 1.6 remains true, with minor changes in the proof, if the function h is just defined on a subset of X.

COROLLARY 2.7: *Let X be a Banach space and let $h: X \to \mathbb{R}$ be a uniformly continuous function such that for every $r \in \mathbb{R}$ the sets $h^{-1}(-\infty, r)$ and $h^{-1}(r, +\infty)$ are both $(\mathcal{C}\setminus\mathcal{C})_\sigma$ subsets of X. Then h is a pointwise limit of differences of convex continuous functions.*

Proof: Given any real number r, the decomposition

$$h^{-1}(-\infty, r) = \bigcup_{n=1}^{\infty} h^{-1}(-\infty, r - n^{-1})$$

shows us how to put $h^{-1}(-\infty, r)$ as a strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$ subset of X (positive distances are ensured by the uniform continuity). For the set $h^{-1}(r, +\infty)$ we can find a similar decomposition. ∎

Theorem 1.6 clearly extends the following result due to Cepedello–Boiso [5].

COROLLARY 2.8: *Let X be a Banach space having an equivalent LUR norm. Then any continuous function is the pointwise limit of a sequence of differences of convex continuous functions.*

3. Banach spaces of the first Borel class

In this section we shall study Banach spaces which are $(\mathcal{F} \vee \mathcal{G})_\delta$ subsets into some dual Banach space endowed with the weak* topology. The important particular class of Banach spaces X such that B_X is a $\mathcal{G}_\delta$-subset of $(B_{X^{**}}, w^*)$ was studied by Edgar and Wheeler in [8]. In that case they proved that $X = X_1 \oplus X_2$ (direct sum) where X_1 and its dual X_1^* are separable and X_2 is reflexive. A trivial but useful observation is the following: if X is a subspace of a dual Banach space Y^*, then $Y^* \setminus X$ is a countable union of $\mathcal{F} \wedge \mathcal{G}$-sets (resp. differences of w^*-compact convex sets, differences of w^*-compact symmetric convex sets) if and only if $B_{Y^*} \setminus B_X$ is a countable union of sets of the same kind.

LEMMA 3.1: *If a topological space V is homeomorphic to a $(\mathcal{F} \vee \mathcal{G})_\delta$ subset of a compact Hausdorff space, then V is hereditarily Baire.*

Proof: Suppose that $V \subset K$ and $V = \bigcap_{n=1}^{\infty}(F_n \cup G_n)$ where F_n is closed and G_n is open in K for every $n \in \mathbb{N}$. It is enough to show that V is Baire since any closed subset of V is also a $(\mathcal{F} \vee \mathcal{G})_\delta$ subset of K. We may assume that V is dense in K. Let F_n^o denote the interior of F_n. It is easy to verify that $F_n^o \cup G_n$ is dense in K. The intersection $\bigcap_{n=1}^{\infty}(F_n^o \cup G_n)$ is a dense Čech-complete subset of V, and thus V is a Baire space. ∎

Proof of Theorem 1.3: The space $(B_X, \sigma(X, Y))$ is Čech-analytic (see [14]) since B_X is a $(\mathcal{F} \vee \mathcal{G})_\delta$ subset of (B_{Y^*}, w^*) and thus it is also hereditarily Baire by the former lemma. By [14, Theorem 4.1], a Čech-analytic space with a finer lower semicontinuous metric d such that its compact subsets are fragmentable by d is itself σ-fragmentable (see [14]) by d. A hereditarily Baire topological space which is σ-fragmentable by a lower semicontinuous metric d is fragmentable by d, [14, Corollary 3.1.2]. If any hereditarily Baire space is fragmented by a metric, then the identity map to the space endowed with the metric has the point of continuity property by [21, Lemma 1.1]. ∎

Remark 3.2: The $\sigma(X, Y)$-compact subsets of a Banach space X are fragmented by the norm if X has an equivalent $\sigma(X, Y)$-Kadec norm, that is, on its unit sphere the norm topology and $\sigma(X, Y)$ coincide [14]. If the dual unit ball (B_{X^*}, w^*) is a Corson compact (for instance, if X is WCD), then the $\sigma(X, Y)$-compact subsets of X are fragmented by the norm for any norming subspace $Y \subset X^*$ [4].

Proof of Theorem 1.4: Let (Ω, Σ, μ) a probability space and let $\nu\colon \Sigma \to X$ be a μ-continuous vector measure with average range in B_X. There exists a w^*-Borel measurable density $f\colon \Omega \to B_{Y^*}$, that is, $\langle \nu(C), y\rangle = \int_C \langle f, y\rangle d\mu$ for every $C \in \Sigma$ and $y \in Y$. The proof is based on the theory of liftings. We shall sketch the proof because we found no suitable reference. For any $y \in Y$, the signed measure $\langle \nu, y\rangle$ is μ-continuous, so it has a Radon–Nikodým derivative $f_y \in L^1(\mu)$. Let ρ be a lifting of $L^\infty(\mu)$; see [1]. It is easy to check that the map $y \to \rho(f_y)(\omega)$ is linear for every $\omega \in \Omega$ and bounded by $\|y\|$, so there is $y_\omega^* \in B_{Y^*}$ such that $y_\omega^*(y) = \rho(f_y)(\omega)$. Clearly, the map defined by $f(\omega) = y_\omega^*$ is w^*-scalarly measurable, so it is w^*-Baire measurable by [7, Theorem 2.3]. We claim that f also is w^*-Borel measurable and the measure image $f(\mu)$ is w^*-Radon. Indeed, if ρ_K is the abstract lifting considered in [1, §2] for the compact space $K = B_{Y^*}$, then

$$h \circ \rho_K(f)(\omega) = \rho(h \circ f)(\omega)$$

for every $\omega \in \Omega$ and every $h \in C(K)$. From the definition of f we get that $\rho_K(f) = f$ just taking as continuous functions h the elements $y \in Y$. The desired properties follow from [1, Theorem 2.1].

Put $B_{Y^*} \setminus B_X = \bigcup_{n=1}^{\infty}(A_n \setminus B_n)$ and observe that B_X is a w^*-Borel subset of B_{Y^*}. We want to show that $f(\omega) \in B_X$ for μ-almost all $\omega \in \Omega$. That is equivalent to showing that $f(\mu)(B_X) = 1$. Fix $n \in \mathbb{N}$ and suppose that $f(\mu)(A_n \setminus B_n) > 0$. Using the fact that $f(\mu)$ is w^*-Radon, we can find a w^*-open halfspace H disjoint with B_n such that $f(\mu)(A_n \cap H) > 0$. Without loss of generality, assume that $L \cap X = \emptyset$, where $L = \overline{A_n \cap H}^{w^*}$ is w^*-compact convex. Let $C = f^{-1}(L)$ and $x = \nu(C)/\mu(C) \in X$. Take $y \in Y$ such that $\langle x, y\rangle < \inf_{y^* \in L}\langle y^*, y\rangle$. Thus

$$\langle x, y\rangle < \mu(C)^{-1} \int_C \langle f, y\rangle d\mu$$

which is a contradiction.

We have now that $f(\mu)$ is a Radon measure on $(B_X, \sigma(X, Y))$. Since the $\sigma(X, Y)$-compact subsets of X are fragmentable, $f(\mu)$ is concentrated on a norm separable subset of B_X. By [7, Theorem 5.2], there is a Bochner measurable function $g\colon \Omega \to B_X$, such that for every $y \in Y$, the equality $\langle g(\omega), y\rangle = \langle f(\omega), y\rangle$ holds for μ-almost all $\omega \in \Omega$. This implies $\langle \nu(C), y\rangle = \langle \int_C g d\mu, y\rangle$ for every $C \in \Sigma$ and $y \in Y$, and thus $\nu(C) = \int_C g d\mu$. This proves the RNP of X. ∎

To provide examples of Banach spaces being weak* first Borel class subsets into some dual Banach space, we need two definitions which are based on the Szlenk index [16].

Definition 3.3: We say that X has countable Y-fragmentability index if for every $\varepsilon > 0$ there exists a decreasing transfinite sequence $(C_\alpha)_{\alpha<\gamma_\varepsilon}$ of $\sigma(X, Y)$-closed subsets of B_X, where γ_ε is countable, such that $B_X = \bigcup_{\alpha<\gamma_\varepsilon}(C_\alpha \setminus C_{\alpha+1})$ and for every $x \in C_\alpha \setminus C_{\alpha+1}$ there is a $\sigma(X, Y)$-open neighbourhood U of x with $\operatorname{diam}(C_\alpha \cap U) < \varepsilon$.

Clearly, if the Banach space X has countable Y-fragmentability index, then B_X endowed with the $\sigma(X, Y)$-topology is norm fragmentable. The converse is true for X a separable Banach space.

Definition 3.4: We say that X has countable Y-dentability index if for every $\varepsilon > 0$ there exists a decreasing transfinite sequence $(C_\alpha)_{\alpha<\gamma_\varepsilon}$ of $\sigma(X, Y)$-closed convex subsets of B_X, where γ_ε is countable, such that $B_X = \bigcup_{\alpha<\gamma_\varepsilon}(C_\alpha \setminus C_{\alpha+1})$ and for every $x \in C_\alpha \setminus C_{\alpha+1}$ there is a $\sigma(X,Y)$-open halfspace H containing x with $\mathrm{diam}(C_\alpha \cap H) < \varepsilon$.

Lancien [16, 17] has studied Banach spaces with countable indices. Among other results, he proved that a dual Banach space X^* has countable X-dentability index if it has countable X-fragmentability index.

THEOREM 3.5: *Let X be a Banach space, $Y \subset X^*$ a norming subspace.*

(a) *If X has countable Y-fragmentability index, then X is a $(\mathcal{F} \vee \mathcal{G})_\delta$ subset of Y^* with respect to the weak* topology.*
(b) *If X has countable Y-dentability index, then $Y^* \setminus X$ is a $(\mathcal{C} \setminus \mathcal{C})_\sigma$ subset of Y^* with respect to the weak* topology.*

Proof: Given $n \in \mathbb{N}$, let $(C^n_\alpha)_{\alpha<\gamma_n}$ be a family of sets like in Definition 3.3 with $\varepsilon = n^{-1}$. For every α we can put

$$C^n_{\alpha+1} = C^n_\alpha \setminus \bigcup_{i\in I^n_\alpha} U^n_i$$

where the family $\{U^n_i : i \in I^n_\alpha\}$ consists of w^*-open subsets of Y^* such that $\mathrm{diam}(C^n_\alpha \cap U^n_i) < n^{-1}$. The set $U^n_\alpha = \bigcup_{\beta<\alpha}\bigcup_{i\in I^n_\beta} U_i$ is w^*-open. Take

$$A = B_{Y^*} \cap \bigcap_{n\in\mathbb{N}} \bigcap_{\alpha<\gamma_n} (\overline{C^n_\alpha}^{w^*} \cup U^n_\alpha)$$

which is clearly $(\mathcal{F} \vee \mathcal{G})_\delta$ in (Y^*, w^*). Clearly $B_X \subset A$. We claim that $B_X = A$. Indeed, take any $z \in A$ and fix $n \in \mathbb{N}$. Since $(\overline{C^n_\alpha}^{w^*})$ is decreasing in α, we can take α_n such that $z \in \overline{C^n_{\alpha_n}}^{w^*}$ but $z \notin \overline{C^n_{\alpha_n+1}}^{w^*}$. This means that $z \in U^n_{\alpha_n}$ and so $z \in U^n_{i_n}$ for some $i_n \in I^n_{\alpha_n}$. We have now that $z \in \overline{C^n_{\alpha_n}}^{w^*} \cap U^n_{i_n}$. Since $\overline{C^n_{\alpha_n}}^{w^*} \cap U^n_{i_n} \subset \overline{C^n_{\alpha_n} \cap U^n_{i_n}}^{w^*}$ and $\mathrm{diam}(\overline{C^n_{\alpha_n} \cap U^n_{i_n}}^{w^*}) \leq n^{-1}$ by the w^*-lower semicontinuity of the norm, so we have $d(z, X) \leq n^{-1}$. Since that happens for every $n \in \mathbb{N}$, then $z \in X$.

Assume now that X has countable Y-dentability index. Then the w^*-open sets U_i above can be taken to be w^*-open halfspaces and it is easy to see that the sets (C^n_α) can be taken to be symmetric. Let us change the notation putting $H_i = U_i$. We have

$$B_X = \bigcap_{n\in\mathbb{N}} \bigcap_{\alpha<\gamma_n} \left(\overline{C^n_\alpha}^{w^*} \cup \bigcup_{\beta<\alpha} \bigcup_{i\in I^n_\beta} H_i\right),$$

thus

$$B_{Y^*} \setminus B_X = \bigcup_{n\in\mathbb{N}} \bigcup_{\alpha<\gamma_n} (D_n \setminus \overline{C^n_\alpha}^{w^*})$$

where $D^n_\alpha = \bigcap_{\beta<\alpha} \bigcap_{i\in I^n_\beta} (Y^* \setminus H_i)$ is w^*-compact and convex. ∎

If X is a separable Banach space with the PCP (resp. RNP) then X has countable X^*-fragmentability (resp. X^*-dentability) index. The following result due to Edgar and Wheeler [8] follows from Theorems 1.3 and 3.5.

COROLLARY 3.6: *Let X be a separable Banach space. Then X has the PCP if and only if X is a $(\mathcal{F} \vee \mathcal{G})_\delta$ subset of X^{**} with respect to the weak* topology.*

Proof of Theorem 1.1: Just apply Theorems 1.4 and 3.5. ∎

The convex counterpart of $\mathcal{G}_\delta$-sets was introduced in [9, 11] as follows. Let $C \subset D$ be subsets of a dual Banach space Y^*. We say that C is a $\mathcal{H}_\delta$ **subset** (resp. **strong** $\mathcal{H}_\delta$ **subset**) of D if $D \setminus C = \bigcup_{n=1}^\infty K_n$ where the sets K_n are w^*-compact and convex (resp. and $d(C, K_n) > 0$ for every $n \in \mathbb{N}$). Clearly, if C is an $\mathcal{H}_\delta$ set then $Y^* \setminus C$ is a $(\mathcal{C} \setminus \mathcal{C})_\sigma$ set. It is not difficult to show that the converse is true for Y separable. Analogous results hold adding "strong".

4. Strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$ remainders and the ANP

In this section we shall discuss some links between the representation of a Banach space X into a dual and renorming properties.

LEMMA 4.1: *Let X be a Banach space and let $Y \subset X^*$ be a norming subspace. The following are equivalent:*

(i) *$Y^* \setminus X$ is a strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$ subset of Y^* with respect to the weak* topology.*
(ii) *There is a sequence (A_n) of convex w^*-compact subsets with nonempty norm interior such that for every $y^* \in Y^* \setminus X$ there is a w^*-open halfspace H such that $y^* \in A_n \cap H$ and $A_n \cap H \subset Y^* \setminus X$.*
(iii) *There are sequences of convex w^*-compact sets (A_n), (B_n) with nonempty norm interior such that $Y^* \setminus X = \bigcup_{n=1}^\infty (A_n \setminus B_n)$.*

Proof: Implications (i) ⇒ (ii) ⇔ (iii) can be deduced from Lemma 2.3 and the ideas of its proof. It is enough to show that statement (ii) above implies statement (iii) of Lemma 2.3, namely we shall prove that the sequence (A_n) given by (ii) verifies that for any $y^* \in Y^* \setminus X$ there is a w^*-open halfspace H such that $y^* \in A_n \cap H$ and $d(A_n \cap H, X) > 0$. Let $A \subset Y^*$ be a w^*-compact convex subset with nonempty norm interior. Suppose that $y^* \in A$ is such that it is contained in a w^*-open halfspace H with $A \cap H \subset Y^* \setminus X$. Take $x \in A^o \cap X$. Since $X = X - x$, we may change A by $A - x$, and without loss of generality assume that 0 is an interior point of A. In that case $y^* \neq 0$. If $H = \{u^* \in Y^*: u^*(y) > a\}$, then take $b > a$ such that the halfspace $H' = \{u^* \in Y^*: u^*(y) > b\}$ still contains y^*. Since $A \cap X \subset A \setminus H$, then $d(A \cap H', A \cap X) > 0$. Take $\varepsilon > 0$ small enough to have

$$d(A \cap H', (1+\varepsilon)(A \cap X)) > 0.$$

Since 0 is an interior point of A, then

$$d(A \cap H', Y^* \setminus (1+\varepsilon)A) > 0.$$

The last two inequalities imply that $d(A \cap H', X) > 0$. ∎

Remark 4.2: Let X be a Banach space with countable Y-dentability index and the index and the sets $(C_\alpha)_{\alpha<\gamma_\varepsilon}$ of Definition 3.4 can be taken with nonempty norm interior. Then the proof of Theorem 3.5 together with Lemma 4.1 gives that $Y^* \setminus X$ is a strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$ subset of Y^* with respect to the weak* topology.

PROPOSITION 4.3: *Let X be a Banach space and $Y \subset X$ a norming subspace. Then the following statements are equivalent:*

(i) *$Y^* \setminus X$ is a strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$ subset of Y^* with respect to the weak* topology.*

(ii) *There is an equivalent dual norm $\|.\|$ on Y^* such that for any $(x_n) \subset X$ and $y^* \in Y^*$ with $\lim_n \|x_n\| = \|y^*\|$ and $\lim_n \|x_n + y^*\| = 2\|y^*\|$, then $y^* \in X$.*

(iii) *There is an equivalent dual norm $\|.\|$ on Y^* such that $S_{Y^*} \cap X$ is relatively weak* closed in S_{Y^*}.*

Proof: (i) $\Rightarrow$ (ii) Follows from Theorem 2.4.

(ii) $\Rightarrow$ (iii) Consider Y^* endowed with a norm satisfying (ii). It is enough to show that $y^* \in X$, if $\|y^*\| = 1$ and $(x_\omega) \subset S_X$ is a net w^*-converging to y^*. Indeed, the net $(x_\omega + y^*)$ is w^*-converging to $2y^*$. Since the norm is w^*-lower semicontinuous we have $\lim_\omega \|x_\omega + y^*\| = 2$. Thus we can take an increasing sequence (ω_n) such that $\lim_n \|x_{\omega_n} + y^*\| = 2$, and so $y^* \in X$.

(iii) $\Rightarrow$ (i) First we claim that for any $y^* \in S_{Y^*} \setminus S_X$, there is a w^*-open halfspace H such that $y^* \in H$ and $B_{Y^*} \cap H \subset Y^* \setminus X$. To see that, just remark that $y^* \notin \overline{B_X}^{w^*}$ and the existence of H is a consequence of the Hahn–Banach theorem. Now take any $y^* \in Y^* \setminus X$. By homogeneity and a small perturbation, there is a rational $r \geq \|y^*\|$ and a w^*-open halfspace H containing y^* such that $rB_{Y^*} \cap H \subset Y^* \setminus X$. Let (A_n) be a reenumeration of the sets of the form rB_{Y^*} with $r \geq 0$ rational. We have proved that for any $y^* \in Y^* \setminus X$, there is $n \in \mathbb{N}$

and H a w^*-open halfspace containing y^* such that $A_n \cap H \subset Y^* \setminus X$. Then the conclusion follows from Lemma 4.1, because the sets A_n have nonempty norm interior. ∎

The following result implies that $\ell_1(\Gamma)$ is a $(\mathcal{F} \vee \mathcal{G})_\delta$, its bidual, which improves the estimate $(\mathcal{F} \wedge \mathcal{G})_{\sigma\delta}$ given in the proof of [7, Theorem 3.3].

THEOREM 4.4: *Assume that X is Banach space which is isomorphic to a dual space with a weak*-Kadec norm. Then $X^{**} \setminus X$ is a strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$ subset of X^{**} with respect to the weak* topology.*

Proof: Let Y be such that $X = Y^*$. Without loss of generality we may assume that the norm of Y^* is w^*-Kadec. Consider in X^* the symmetric convex body $\overline{conv}^{\|\cdot\|}(B_Y \cup 2^{-1}B_{Y^{**}})$ which is the unit ball of some equivalent (but non-dual) norm on X^*. Let $|||.|||$ be dual norm, defined on X^{**}. It is easy to verify that if we endow X^{**} with $|||.|||$, then $S_{X^{**}} \cap X = S_X$. We shall prove that S_X is a closed subset of $S_{X^{**}}$ and the result will follow from Proposition 4.3. Let $(x_\omega) \subset S_X$ be a net which is w^*-converging to $x^{**} \in S_{X^{**}}$. Since $x^{**} \in B_{Y^{***}}$, its supremum on $2^{-1}B_{Y^{**}}$ is less than 2^{-1}. But $|||x^{**}||| = 1$, so the supremum of x^{**} on B_Y must be 1. Put $x = x^{**}|_Y$ and realize that $x \in S_{Y^*}$. Clearly (x_ω) converges to x in the weak* topology of Y^* and $\|x_\omega\| = \|x\|$. Since the norm is w^*-Kadec, the net (x_ω) is norm converging to x. This implies that $x^{**} = x$ and thus $x^{**} \in S_X$. ∎

Recall that ANP along this paper means ANP-III in the terminology of [13]. We shall use a nice characterization from [12] of the ANP to prove the following.

PROPOSITION 4.5: *A Banach space X has the ANP if and only if there is an equivalent norm on X^* such that its dual norm on X^{**} induces the original norm of X, that is $S_X = S_{X^{**}} \cap X$, and S_X relatively w^*-closed in $S_{X^{**}}$.*

Proof: Assume that X has the Φ-ANP for some norming subset $\Phi \subset B_{X^*}$. It is easy to see that if a sequence $(x_n) \subset S_X$ is asymptotically normed by $\overline{conv}^{\|\cdot\|}(\Phi \cup -\Phi \cup 2^{-1}B_{X^*})$, then (x_n) has a subsequence (x_{n_k}) such that (x_{n_k}) or $(-x_{n_k})$ is asymptotically normed by Φ. Without loss of generality we may replace Φ by that convex set, which is the unit ball of some equivalent norm on X^*. Call $|||.|||$ that norm on X^* and notice that once X^{**} is endowed with the dual norm, then $S_{X^{**}} \cap X = S_X$. In [12] it is proved that if X has the Φ-ANP for any $x^{**} \in X^{**} \setminus X$, then $\sup_{\phi \in \Phi} x^{**}(\phi) < \|x^{**}\|$. It follows that $S_X = \overline{B_X}^{w^*} \cap S_{X^{**}}$, so S_X is relatively w^*-closed in $S_{X^{**}}$. Reciprocally, if $S_X = S_{X^{**}} \cap X$ and S_X

relatively w^*-closed in $S_{X^{**}}$, then X^{**} is endowed with the dual of some norm $|\!|\!|.|\!|\!|$ on X^*. Let $\Phi = B_{X^*}$ and consider a Φ-asymptotically normed sequence (x_n). Clearly, any w^*-cluster point of (x_n) in X^{**} must be in $S_{X^{**}}$, and thus in X by the hypothesis. This implies that (x_n) is a relatively weakly compact subset and so $\bigcap_{n=1}^{\infty} \overline{conv}^{\|\cdot\|}(\{x_m : m \geq n\}) \neq \emptyset$. ∎

Combining the last result with Proposition 4.3 we obtain the following result which contains Theorem 1.5.

THEOREM 4.6: *For a Banach space X the following are equivalent:*

(i) *X has the ANP with some equivalent norm.*

(ii) *$X^{**}\backslash X$ is a strong $(\mathcal{C}\backslash\mathcal{C})_\sigma$ subset of X^{**} with respect to the weak* topology.*

(iii) *There is an equivalent dual norm $\|.\|$ on X^{**} such that for any $(x_n) \subset X$ and $x^{**} \in X^{**}$ with $\lim_n \|x_n\| = \|x^{**}\|$ and $\lim_n \|x_n + x^{**}\| = 2\|x^{**}\|$, then $x^{**} \in X$.*

As an application, we shall prove that separable spaces with the RNP can be renormed with a property stronger than the ANP.

PROPOSITION 4.7: *Let X be a separable Banach space with the RNP. Then there is an equivalent dual norm $\|.\|$ on X^{**} such that for any $(x_n) \subset X$ and $x^{**} \in X^{**}$ with $\lim_n \|x_n\| = \|x^{**}\|$ and $\lim_n \|x_n + x^{**}\| = 2\|x^{**}\|$, then $\lim_n \|x_n - x^{**}\| = 0$.*

Proof: Since X is separable, there is an equivalent dual norm $\|.\|_1$ on X^{**} such that its restriction to X is LUR. Suppose that $X^{**}\backslash X$ is a strong $(\mathcal{C}\backslash\mathcal{C})_\sigma$ subset of X^{**} and let $\|.\|_2$ be the norm on X^{**} given by (iii) of Theorem 4.6. We claim that the norm $\|.\|$ defined by $\|.\|^2 = \|.\|_1^2 + \|.\|_2^2$ satisfies the required condition. Indeed, if $(x_n) \subset X$ is a sequence with $\lim_n \|x_n\| = \|x^{**}\|$ and $\lim_n \|x_n + x^{**}\| = 2\|x^{**}\|$, then by [6, Fact 2.3], we have $\lim_n \|x_n\|_i = \|x^{**}\|_i$ and $\lim_n \|x_n + x^{**}\|_i = 2\|x^{**}\|_i$, for $i = 1, 2$. Taking $i = 2$, we deduce that $x^{**} \in X$, and taking $i = 1$, we obtain $\lim_n \|x_n - x^{**}\| = 0$ by the LUR property.

We shall prove that $X^{**} \setminus X$ is a strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$ subset of X^{**}. According to Remark 4.2 we just need to build sets $(C_\alpha)_{\alpha<\gamma_\varepsilon}$ as in Definition 3.4 with nonempty norm interior. We shall use the following fact: if X is a Banach space with the RNP and $C, D \subset X$ closed convex subsets such that $C \setminus D \neq \emptyset$ and $\varepsilon > 0$, then there is an open halfspace H such that $H \cap C \neq \emptyset$, $D \cap H = \emptyset$ and $\operatorname{diam}(C \cap H) < \varepsilon$. That is a consequence of the norm density in X^* of the elements x^* strongly exposing points of C [3, Theorem 3.5.4]. Fix $\varepsilon > 0$ and define a transfinite sequence of closed convex sets $(C_\alpha)_{\alpha \leq \gamma}$ which is strictly decreasing and such that $C_1 = B_X$, $C_\gamma = 2^{-1}B_X$, $C_\alpha = \bigcap_{\beta<\alpha} C_\beta$ if α is a limit ordinal and

$C_{\alpha+1} = C_\alpha \setminus H_\alpha$, where $H_\alpha = \{x \in X : x^*_\alpha(x) > s_\alpha\}$ is an open halfspace disjoint from $2^{-1}B_X$ such that $\operatorname{diam}(C_\alpha \cap H_\alpha) < \varepsilon$. The construction is possible because of the fact and γ must be countable since (C_α) is strictly decreasing and X is separable. To obtain a sequence $(C_\alpha)_{\alpha<\gamma_\varepsilon}$ as in Definition 3.4 we have to repeat the process again starting at $2^{-1}B_X$ and finishing at $2^{-2}B_X$, and so on. Clearly, it is enough to iterate that process n times, where n is such that $2^{-n+1} < \varepsilon$, to reach the empty set with some C_α. Then take γ_ε as the first index α such that $C_\alpha = \emptyset$. ∎

It is possible to arrange the ideas behind Proposition 4.7 in order to give a quite self-contained proof of the result of Ghoussoub and Maurey [10] on the equivalence of the RNP and the ANP for separable Banach spaces. The key fact, as above, is "to eat" the unit ball in such a way that the remainders have nonempty interior.

We do not know if the ANP implies the existence of an equivalent Kadec norm. A negative answer will provide a Banach space with the RNP and no equivalent LUR norm. Actually, a Banach space with RNP and a Kadec norm has an equivalent LUR norm [20]. As a consequence, if a Banach space has the ANP and an equivalent Kadec norm, then X can be renormed to verify stronger asymptotic-norming properties [12]. Recall that a Banach space with countable dentability index has an equivalent LUR norm [16]. In fact, if the sets $(C_\alpha)_{\alpha<\gamma_\varepsilon}$ of Definition 3.4 have nonempty norm interior, a convex series of their Minkowski functionals produces a norm which is LUR and has the ANP; see Remark 4.2.

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